NONLINEAR INTERACTION OF HIGH-FREQUENCY AND LOW-FREQUENCY WAVES

V. E. Zakharov and A. M. Rubenchik UDC 533.952;534.2

The interaction of high-frequency waves with low-frequency (acoustic) waves is investigated. The analysis is carried out in the Hamiltonian formalism in the interest of generality. The instability problem is investigated for the high-frequency wave. The general results obtained in the article are applied to the stability analysis of electromagnetic waves in plasmas and dielectrics. Wave propagation in weakly dispersive media is considered. It is shown that the waves are unstable. The possibility of self-focusing of the waves is studied.

Wave coupling in nonlinear media frequently involves the participation of two types of waves: high-frequency (HF) and low-frequency (LF) (the low-frequency wave is an acoustic wave in the majority of cases). This kind of situation occurs, for example, in the interaction of electromagnetic and acoustic waves in a dielectric, of spin and elastic waves in a ferromagnet, and of various wave modes in a plasma.

The interaction of HF and LF waves in various media is amenable to treatment from a unified point of view. Such an attempt has been made by Rudakov and Vedenov [1] on the basis of the Lagrangian formalism for a nonlinear medium (see also [2, 3]). In [1], however, the geometric-optical approximation is used for the HF wave, and in many respects this approximation is inadequate.

In the present article we investigate the interaction of HF and LF waves in an arbitrary medium for which the Hamiltonian is the energy expressed in canonical variables.

This statement of the problem turns out to be fairly comprehensive insofar as canonical variables can be introduced for many media, covering the two-fluid hydrodynamical model of a plasma [4], magnetohydrodynamics [5], waves on the surface of a fluid [6], and ferromagnets [7, 8]. The introduction of canonical variables makes it possible to apply a unified treatment to the problem of the instability of a strong HF wave in a medium in which it is possible for LF waves to propagate.

Special cases of this instability have been investigated before, including the decay instability of a Langmuir wave [9], stimulated Mandel'shtam–Brillouin scattering (SMBS) [10, 11], and the "electroacoustic" instability of an electromagnetic wave in a plasma [12-15]. In the present study we show (Sec. 1) that the interaction Hamiltonian for the interaction of a HF wave of not too large amplitude with LF waves has a simple form, such that within the context of this Hamiltonian the HF instability problem has an exact solution. The simplest instabilities are those which result in the growth of HF waves scattered at large angles. Such instabilities are the analog of SMBS and actually go over to SMBS for small-amplitude HF waves in the case of electromagnetic waves in a dielectric. Like SMBS, they can grow only for long coherent wave packets. With a reduction in the scattering angle the instability acquires another character and is preserved for a stationary wave with a random phase angle. This instability results in the growth of transverse inhomogeneities of the medium and subsequently to "disintegration" of the HF waves, i.e., the formation of local amplitude singularities. Analogous effects are induced by the "intrinsic" nonlinearity of HF waves (a field-squared correction to the refractive index for electromagnetic waves), which is included in our discussion from the outset.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 84-98, September-October, 1972. Original article submitted September 3, 1971.

© 1974 Consultants Bureau, a division of Plenum Publishing Corporation, 227 West 17th Street, New York, N. Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.

In Sec. 4 we apply the results of the general theory to the interaction of electromagnetic and acoustic waves in plasmas and dielectrics. In Sec. 5 we solve the hydrodynamic stability problem for a wave in a medium with weak sound dispersion. Examples of such waves are ion-acoustic waves in a plasma and waves on the surface of a shallow liquid. We show that these waves are stable in the one-dimensional problem and unstable in the multidimensional case.

1. The Fundamental Model

We consider the interaction of HF and LF waves in a conservative medium characterized by a Hamiltonian H. We introduce the complex normal amplitudes a_k for HF and b_k for LF waves. The quadratic part of the Hamiltonian of the medium has the form

$$H_0 = \int \omega_k a_k a_k^* \, d\mathbf{k} + \int \Omega_k b_k b_k^* \, d\mathbf{k}$$

where ω_k and Ω_k are the dispersion laws for HF and LF waves. The wave interaction Hamiltonian can be written in the form

$$H_i = H_i^{(1)} + H_i^{(2)} + H_i^{(3)}$$

Here $H_i^{(1)}$ describes the mutual interaction of HF waves, $H_i^{(2)}$ the interaction of HF and LF waves, and $H_i^{(3)}$ the mutual interaction of LF waves. Equations for $a_\mathbf{k}$ and $b_\mathbf{k}$ can be obtained by variation of the Hamiltonian $H = H_0 + H_i$ according to the rule

$$\frac{\partial a_\mathbf{k}}{\partial t} = -i \frac{\delta H}{\delta a_\mathbf{k}^*}, \quad \frac{\partial b_\mathbf{k}}{\partial t} = -i \frac{\delta H}{\delta b_\mathbf{k}^*} \tag{1.1}$$

In the linear approximation the wave amplitudes have the time dependences

$$a_\mathbf{k} = a_\mathbf{k}^{(0)} \exp[-i\omega(k_0)t]$$
$$b_\mathbf{k} = b_\mathbf{k}^{(0)} \exp[-i\Omega(k_0)t]$$

We assume hereinafter that the LF wave amplitude is small ($b_k \ll a_k$), whereupon we neglect the LF mutual interaction Hamiltonian $H_i^{(3)}$ and in $H_i^{(2)}$ retain only terms linear in b_k. Also, we limit the discussion to the lowest-order terms in a_k that are nonvanishing under averaging over the HF period. These requirements enable us to find the Hamiltonian

$$H_i^{(2)} = \int [\Gamma_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} b_\mathbf{k} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} + (\ )^*] \delta_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2} \, d\mathbf{k} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \tag{1.2}$$

Here ( )* denotes the complex conjugate.

We note that the theory is valid for the case in which

$$H_i^{(2)} \gg \int \Omega_\mathbf{k} b_\mathbf{k} b_\mathbf{k} \, d\mathbf{k}$$

and the HF waves cause strong "retuning" of the LF waves. We choose the Hamiltonian $H_i^{(1)}$ in the form

$$H_i^{(1)} = \frac{1}{2} \int W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a_\mathbf{k}^* a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} a_{\mathbf{k}_3} \delta_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} \, d\mathbf{k} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3 \tag{1.3}$$

This form of $H_i^{(1)}$ is realized in a medium with a cubic nonlinearity as well as in certain cases of square-law nonlinear media (see Sec. 5). The equations of motion (1.1) have the form

$$\begin{aligned} \frac{\partial a_k}{\partial t} + i\omega_k a_k &= -i \int (\Gamma_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2} b_{\mathbf{k}_1} + \Gamma_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}}^* b_{-\mathbf{k}_1}^*) a_{\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \, d\mathbf{k}_1 \, d\mathbf{k}_2 - \\ &- i \int W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} a_{\mathbf{k}_3} \delta_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3 \\ \frac{\partial b_\mathbf{k}}{\partial t} + i\Omega_\mathbf{k} b_\mathbf{k} &= -i \int \Gamma_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^* a_{\mathbf{k}_1} a_{\mathbf{k}_2}^* \delta_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \end{aligned} \tag{1.4}$$

The interaction Hamiltonian is greatly simplified if the HF waves form a narrow packet near $k \sim k_0$ in k-space. We can then put

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \approx W_{\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0} = q$$
$$\Gamma_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \approx \Gamma_{\mathbf{k}\mathbf{k}_0\mathbf{k}_0} = f(\mathbf{k}, \mathbf{k}_0)$$
$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + \frac{\partial\omega}{\partial\mathbf{k}} \delta\mathbf{k} + \frac{1}{2} \frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta} \delta k_\alpha \delta k_\beta$$

In the event that the LF waves are acoustic, $\Omega = sk$, we can calculate the function $f(\mathbf{k}, \mathbf{k}_0)$ and the complete Hamiltonian of the system in explicit form. To do so we transform from the amplitude a_k to the following:

$$\Psi(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int a_{\mathbf{k}} \exp(i\omega(k_0)t + i(\mathbf{k} - \mathbf{k}_0)r)\, d\mathbf{k} \tag{1.5}$$

and from the amplitude b_k to the density variation $\delta\rho$ and velocity v of the medium:

$$\delta\rho = \frac{1}{(2\pi)^{3/2}} \int \frac{\rho_0^{1/2} k^{1/2}}{\sqrt{2}\, s^{1/2}} (b_{\mathbf{k}} + b_{-\mathbf{k}}^*) e^{-i\mathbf{k}\mathbf{r}}\, d\mathbf{k} \tag{1.6}$$

$$\mathbf{v} = -\frac{i}{(2\pi)^{3/2}} \int \frac{\mathbf{k}}{k^{1/2}} \frac{s^{1/2}}{\sqrt{2}\, \rho_0^{1/2}} (b_{\mathbf{k}} - b_{-\mathbf{k}}^*) e^{i\mathbf{k}\mathbf{r}}\, d\mathbf{k}$$

The energy of a HF packet narrow in k-space is

$$\varepsilon \approx \omega(\mathbf{k}_0) \int a_{\mathbf{k}} a_{\mathbf{k}}^*\, d\mathbf{k}$$

and its local density is $\omega(\mathbf{k}_0)|\Psi(\mathbf{r}, t)|^2$.

In the presence of an acoustic wave the quantity $\omega(\mathbf{k}_0)$ acquires the variation

$$\delta\omega(\mathbf{k}_0) = \frac{\partial\omega(\mathbf{k}_0)}{\partial\rho}\delta\rho + \frac{\partial\omega(\mathbf{k}_0)}{\partial\mathbf{v}}\mathbf{v}$$

and the corresponding variation of the HF wave energy is

$$\delta\varepsilon = \int |\Psi|^2 \left(\frac{\partial\omega}{\partial\rho}\delta\rho + \frac{\partial\omega}{\partial\mathbf{v}}\mathbf{v}\right) d\mathbf{r} \tag{1.7}$$

The quantity $\delta\varepsilon$ clearly coincides with $H_i^{(2)}$. In an isotropic medium

$$\frac{\partial\omega(\mathbf{k}_0)}{\partial\mathbf{v}} = \alpha\frac{\mathbf{k}_0}{|\mathbf{k}_0|}$$

and we denote

$$\frac{\partial\omega(k_0)}{\partial\rho} = \beta$$

Moreover, we can put $\mathbf{v} = \nabla\Phi$, where Φ is the hydrodynamic potential.

In the variables Ψ, $\delta\rho$, and Φ the complete Hamiltonian of the medium has the form

$$H = \int \Big\{\omega_0|\Psi|^2 + \frac{i}{2}\left(\Psi^*\frac{\partial\Psi}{\partial z} - \Psi\frac{\partial\Psi^*}{\partial z}\right)V_g + \frac{1}{2}\omega_k''\left|\frac{\partial\Psi}{\partial z}\right|^2 + \\ + \frac{V_g}{2k_0}|\nabla_\perp\Psi|^2 + \frac{1}{2}q|\Psi|^4 + \frac{1}{2}\rho_0(\nabla\Phi)^2 + \frac{s^2\delta\rho^2}{2\rho_0} + |\Psi|^2\left(\beta\delta\rho + \alpha\frac{\partial\Phi}{\partial z}\right)\Big\}\, d\mathbf{r} \tag{1.8}$$

where V_g is the group velocity.

We have made use here of the fact that

$$\frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta}\frac{\partial^2}{\partial x_\alpha \partial x_\beta} = \omega_k''\frac{\partial^2}{\partial z^2} + \frac{V_g}{k_0}\Delta_\perp \qquad \left(\mathbf{V}_g = \frac{\partial\omega}{\partial\mathbf{k}}\right)$$

in an isotropic medium, where z is the coordinate in the direction of $\mathbf{k}_0$.

It is evident from Eq (1.5) that the quantity Ψ represents a canonical transformation from a_k, so that

$$i\frac{\partial\Psi}{\partial t} = iV_g\frac{\partial\Psi}{\partial z} - \frac{1}{2}\omega_k''\frac{\partial^2\Psi}{\partial z^2} - \frac{V_g}{2k_0}\Delta_\perp\Psi + \\ + \Psi\left(q|\Psi|^2 + \beta\delta\rho + \alpha\frac{\partial\Psi}{\partial z}\right) = \delta\left(H - \int\omega_0|\Psi|^2 d\mathbf{r}\right)(\delta\Psi^*)^{-1} \tag{1.9}$$

The variables $\delta\rho$ and Φ form a pair of canonically conjugate variables (see, e.g., [4]), which are given by equations of the form

$$\frac{\partial\delta\rho}{\partial t} = -\rho_0\Delta\Phi - \alpha\frac{\partial}{\partial z}|\Psi|^2 = \frac{\delta H}{\delta\Phi} \tag{1.10}$$

$$\frac{\partial\Phi}{\partial t} = -s^2\frac{\delta\rho_0}{\rho_0} - \beta|\Psi|^2 = -\frac{\delta H}{\delta\rho} \tag{1.11}$$

Substituting (1.5) and (1.6) into (1.7), we find for $f(\mathbf{k}, \mathbf{k}_0)$

$$f(\mathbf{k}, \mathbf{k}_0) = \left(\frac{k}{16\pi^3 s\rho_0}\right)^{1/2}\left(\beta\rho_0 + \alpha s \frac{(\mathbf{k}, \mathbf{k}_0)}{kk_0}\right) \tag{1.12}$$

Equations (1.9)-(1.11) describe the interaction of any kind of HF waves with acoustic waves. It is apparent that two mechanisms exist for the given interaction: variation of the HF wave frequency due to modulation of the density of the medium by sound and "entrainment" of the HF wave by the moving medium. Inasmuch as

$$\beta = \frac{\partial\omega}{\partial\rho} \sim \frac{\omega}{\rho_0}, \qquad \alpha = \frac{\partial\omega}{\partial v} \sim \frac{\omega}{V_p}$$

where V_p is the HF phase velocity, the ratio of these effects are of the order s/V_p; for $s \ll V_p$ the HF wave entrainment effect can be neglected.

In the stationary case, when $\partial/\partial t = 0$, terms proportional to $\partial^2/\partial z^2$ are neglected in Eqs. (1.9)-(1.11). Upon rejecting them we obtain the nonlinear parabolic equation

$$-2ik_0 \frac{\partial\Psi}{\partial z} + \Delta_\perp \Psi = \frac{2k_0}{V}(q - q')\,|\Psi|^2\,\Psi, \qquad q' = \frac{\beta^2\rho_0}{s^2}$$

which describes stationary self-focusing [16, 17]. Self-focusing takes place if $q - q' < 0$.

2. Stability of a Stationary Wave

As Eq. (1.10) indicates, for the interaction of HF with acoustic waves

$$\Gamma(0, \mathbf{k}, \mathbf{k}) = f(0, \mathbf{k}) = 0$$

Under this condition the set of equations (1.4) has an exact solution:

$$b_{\mathbf{k}} = 0, \quad a_{\mathbf{k}} = Ae^{-i\omega_0 t}\delta_{\mathbf{k}-\mathbf{k}_0}, \quad \omega_0 = \omega_{\mathbf{k}} + q\,|A|^2 \tag{2.1}$$

which represents a stationary monochromatic wave.

Let $a_k = c_k e^{-i\omega_0 t}$ and let us linearize Eq. (1.4) against the background of the solution (2.1), assuming that b_k, b_k^*, c_k, c_k^* depend on the time as $e^{-i\Omega t}$. The linearized problem has an exact solution, and the primary wave turns out to be coupled with a pair of HF waves having wave vectors $\mathbf{k} \pm \mathbf{p}$ and a pair of LF waves having vectors $\pm\mathbf{p}$. We have the following dispersion relation for Ω:

$$\begin{aligned}
&\Big\{\Omega - \omega_{\mathbf{k}_0+\mathbf{p}} + \omega_{\mathbf{k}_0} - 2W_{\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0}|A|^2 + q\,|A|^2 + \\
&+ \frac{(\Gamma^*_{-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0+\mathbf{p}})^2|A|^2}{\Omega_{\mathbf{p}} + \Omega} + \frac{|\Gamma_{\mathbf{p},\,\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0}|^2|A|^2}{\Omega_{\mathbf{p}} - \Omega}\Big\} \times \Big\{-\Omega - \omega_{\mathbf{k}_0-\mathbf{p}} + \\
&+ \omega_{\mathbf{k}_0} - 2W_{\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}|A|^2 + q\,|A|^2 + \frac{|\Gamma_{-\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}|^2|A|^2}{\Omega_{\mathbf{p}} + \Omega} + \\
&+ \frac{(\Gamma^*_{-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0-\mathbf{p}})^2|A|^2}{\Omega_{\mathbf{p}} - \Omega}\Big\} = |A|^4\Big(W_{\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0} + \\
&+ \frac{\Gamma_{\mathbf{p},\,\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0}\Gamma^*_{\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0-\mathbf{p}}}{\Omega_{\mathbf{p}} - \Omega} + \frac{\Gamma^*_{-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0+\mathbf{p}}\Gamma^*_{-\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}}{\Omega_{\mathbf{p}} + \Omega}\Big) \times \\
&\times \Big(W_{\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0} + \frac{\Gamma^*_{\mathbf{p},\,\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0}\Gamma^*_{\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0-\mathbf{p}}}{\Omega_{\mathbf{p}} - \Omega} + \frac{\Gamma^*_{-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0+\mathbf{p}}\Gamma_{-\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}}{\Omega_{\mathbf{p}} + \Omega}\Big)
\end{aligned} \tag{2.2}$$

We analyze the dispersion relation (2.2) in the limit as $|A|^2 \to 0$. Near the surface

$$\omega_{\mathbf{k}_0} = \omega_{\mathbf{k}_0-\mathbf{p}} + \Omega_{\mathbf{p}} \tag{2.3}$$

Eq. (2.2) becomes simplified:

$$(\Omega - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0-\mathbf{p}})(\Omega - \Omega_{\mathbf{p}}) + |A|^2|\Gamma_{\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}|^2 \tag{2.4}$$

and has the solution

$$\Omega = \frac{\omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{p}} + \Omega_{\mathbf{p}}}{2} \pm \sqrt{\frac{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_0-\mathbf{p}} - \Omega_{\mathbf{p}})^2}{4} - |A|^2|\Gamma_{\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}|^2}$$

which is unstable for

$$(\omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{p}} + \Omega_{\mathbf{p}})^2 < 4\,|A|^2\,|\Gamma_{\mathbf{p},\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}|^2$$

This instability represents stimulated Mandel'shtam–Brillouin scattering [10, 11] or a first-order decay instability [9]. Its growth rate has the maximum

$$\gamma_{\max} = |A||\Gamma_{\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}|$$

which is attained on the surface (2.3). We now calculate the group velocity for the perturbation induced by this instability. From (2.4) we have

$$\frac{d\,\mathrm{Re}\,\Omega}{d\mathbf{p}} = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2), \quad \mathbf{v}_1 = \frac{\partial \Omega_{\mathbf{p}}}{\partial \mathbf{p}}, \quad \mathbf{v}_2 = \frac{\partial \omega_{\mathbf{k}_0-\mathbf{p}}}{\partial \mathbf{p}} \tag{2.5}$$

The instability is concomitant in a reference frame moving with the group velocity of the primary wave.

Another type of instability occurs near the surface

$$2\omega_{\mathbf{k}_0} = \omega_{\mathbf{k}_0+\mathbf{p}} + \omega_{\mathbf{k}_0-\mathbf{p}} \tag{2.6}$$

(unless this surface is degenerate at the point $\mathbf{p}=0$).

Let $\mathbf{p}$ be a certain point on the indicated surface. We introduce

$$\mathbf{u}_1 = \frac{\partial \omega_{\mathbf{k}_0-\mathbf{p}}}{\partial \mathbf{p}}, \quad \mathbf{u}_2 = \frac{\partial \omega_{\mathbf{k}_0+\mathbf{p}}}{\partial \mathbf{p}}$$

and the deviation $\delta\mathbf{p}$ from that point. Equation (2.2) can be simplified:

$$(\Omega - (\delta\mathbf{p}\mathbf{u}_1) + Q_1|A|^2)(\Omega - (\delta\mathbf{p}\mathbf{u}_2) - Q_2|A|^2) = -|P^2||A|^4$$

Here

$$Q_1 = -2W_{\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0} + q + \frac{(\overset{*}{\Gamma}_{-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0+\mathbf{p}})^2}{\Omega_{\mathbf{p}} + \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{p}}} + \frac{|\Gamma_{\mathbf{p},\,\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0}|^2}{\Omega_{\mathbf{p}} - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0+\mathbf{p}}},$$

$$Q_2 = -2W_{\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0} + q + \frac{(\Gamma_{\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0})^2}{\Omega_{\mathbf{p}} + \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{p}}} + \frac{|\Gamma_{-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0-\mathbf{p}}|^2}{\Omega_{\mathbf{p}} - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0+\mathbf{p}}}.$$

$$P = W_{\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0} + \frac{\Gamma_{\mathbf{p},\,\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0}\overset{*}{\Gamma}_{-\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}}{\Omega_{\mathbf{p}} - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0-\mathbf{p}}} + \frac{\Gamma_{-\mathbf{p},\,\mathbf{k}_0,\,\mathbf{k}_0+\mathbf{p}}\overset{*}{\Gamma}_{-\mathbf{p},\,\mathbf{k}_0-\mathbf{p},\,\mathbf{k}_0}}{\Omega_{\mathbf{p}} + \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{p}}}$$

The maximum growth rate occurs for

$$\delta p = \frac{|Q_1 + Q_2||A|^2}{u_{1\perp} - u_{2\perp}}, \quad u_{i\perp} = \frac{(\mathbf{u}_i\delta\mathbf{p})}{\delta\mathbf{p}}$$

and is equal to

$$\gamma_{\max} = |P||A|^2$$

The group velocity of the disturbance is

$$\frac{\partial \Omega}{\partial \mathbf{p}} = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2) \tag{2.7}$$

The instability described here represents a second-order decay instability [18], and it occurs in an isotropic medium if $\omega_k'' < 0$.

The foregoing discussion is valid for not too small values of p/k_0; as $p/k_0 \to 0$ it is impossible to separate first- and second-order decay instabilities, so that this case must be analyzed separately.

3. Long-Wave Perturbations

For $p \ll k_0$ it is convenient to investigate the instability of a stationary wave within the framework of the system (1.9)-(1.11). Linearizing the system against the background of the solution

$$\delta\rho = 0, \ \Phi = \beta A^2 t, \qquad \Psi = Ae^{-iq|A|^2 t}$$

we deduce the dispersion relation

$$\left\{(\Omega - pV\cos\theta)^2 - \frac{L^2(\theta)\,p^4}{4}\right\}(\Omega^2 - p^2s^2) =$$
$$= L(\theta)\,p^2A^2\left\{q(\Omega^2 - p^2s^2) + \beta^2\rho_0p^2 + 2\alpha\beta p\Omega\cos\theta + \frac{\alpha^2s^2p^2\cos^2\theta}{\rho_0}\right\} \tag{3.1}$$
$$L(\theta) = \omega''\cos^2\theta + Vk_0^{-1}\sin^2\theta, \quad \cos\theta = (p, k_0)/pk_0$$

This relation has been derived by Gurovich and Karpman [13] for the special case $\alpha = q = \theta = 0$. It can also be obtained from Eq. (2.2) by passing to the limit as $p/k_0 \to 0$.

We next analyze Eq. (3.1) for the special case $V < s$, when first-order decay instability is absent. We can then put $\Omega \approx pV \cos\theta$ and reduce Eq. (3.1) to the form

$$(\Omega - pV\cos\theta)^2 - {}^1/_4 L^2(\theta)p^4 = q_{\text{eff}}(\theta) L(\theta) p^2 A^2$$
$$q_{\text{eff}} = q - \frac{\beta^2\rho_0 + 2\alpha\beta V\cos^2\theta + \alpha^2 s^2 \cos^2\theta \rho_0^{-1}}{s^2 - V^2\cos^2\theta} \tag{3.2}$$

whereupon

$$\Omega - pV\cos\theta = \pm\sqrt{Lp^2 q_{\text{eff}} A^2 + {}^1/_4 L^2 p^4} \tag{3.3}$$

The interaction with LF waves "renormalizes" the intrinsic nonlinearity of the HF waves. Instability occurs if there is an interval of angles in which

$$L(\theta)\, q_{\text{eff}}(\theta) < 0$$

In each direction θ the instability domain is bounded by the values

$$p^2 < 4\,|\, q_{\text{eff}}(\theta)/L(\theta)\,|\, A^2$$

where the maximum growth rate $\gamma = q_{\text{eff}} A^2$ is attained for $p^2 = 2\,|\,q_{\text{eff}}(\theta)/L(\theta)|\,A^2$. If $\omega_k'' < 0$, we have $L(\theta) = 0$ for an angle such that $\tan^2\theta = k|\omega_k''|\,V^{-1}$. In this direction the domain of instability is unbounded and goes over with increasing p to a decay instability of second order.

As we see from Eq. (3.3), the instability described above is absolute in the primary wave system. It causes a buildup of the modulation, which is "at rest" with respect to the primary wave, and it is appropriately called modulation instability. Modulation instability can develop for sufficiently short wave packets and leads to strong disintegration of the primary wave (see [19]). It is evident from Eq. (3.3) that the group velocity is

$$\frac{\partial\Omega}{\partial p} \sim V + \frac{1}{2} p^2 \frac{\partial^3\omega_k}{\partial k^3}$$

The absolute character of the instability in the wave system is preserved up to $p/k \sim (qA^2/kV)^{1/3}$.

We now examine the case $s < V$. Decay instability takes place near the cone $\cos\theta = s/V$. The growth rate of the decay instability as $p/k \to 0$ has the form

$$\gamma_p = f(p, k)\,|\,A\,| \approx \omega_k\,(p/k)^{1/2}\,(q' A^2/\omega_k)^{1/2} \tag{3.4}$$

This equation is valid if

$$\gamma_p \ll Lp^2, \qquad \gamma_p \ll sp$$

Next we limit the problem to the case $s \ll V$.

For not too small values of q ($q/q' \gg s^2/V^2$) the influence of the LF waves can be neglected in the entire range of angles except near $\theta = \pi/2$, and it can be assumed that only intrinsic modulation instability occurs in that same range. For angles close to $\theta = \pi/2$ we can put $L \approx V/k_0$ and neglect terms containing α as well, whereupon we obtain

$$\{(\Omega - pV\cos\theta)^2 - {}^1/_4 V^2 p^4 k^{-2}\}(\Omega^2 - p^2 s^2) = Vp^2 A^2 k^{-1}[(\Omega^2 - p^2 s^2)\,q + q' p^2 s^2] \tag{3.5}$$

In the analysis of (3.5) we initially let $q = 0$. Then the following cases are possible.

1°. $\omega_k^{-1} q' A^2 \ll s^2 V^{-2}$. In this case, for

$$p/k_0 \gg \left(\frac{s}{V}\frac{q' A^2}{\omega_k}\right)^{1/2}$$

a first-order decay instability takes place. For smaller values of p the first of conditions (3.4) is violated. For

$$\frac{p}{k} \ll \left(\frac{s}{V}\frac{q' A^2}{\omega_k}\right)^{1/2}$$

we can neglect the term $(Lp^2)^2$ in Eq. (3.5) and simplify the latter:

$$(\Omega - pV\cos\theta)^2(\Omega - ps) = Vk^{-1}p^3 s q' A^2 \tag{3.6}$$

The strongest instability takes place on the cone $\cos\theta = s/V$, where

$$\operatorname{Im}\Omega = \frac{\sqrt{3}}{2} p \left(\frac{Vsq'A^2}{k}\right)^{1/3} \tag{3.7}$$

Differentiating (3.6) with respect to $\mathbf{p}$, we find up to small terms

$$\frac{\partial\Omega}{\partial\mathbf{p}} = \frac{2}{3}\mathbf{V} \tag{3.8}$$

2°. $S^2/V^2 \ll q'A^2/\omega_k \ll s/V$. Now for

$$p/k \gg \left(\frac{V}{s}\frac{q'A^2}{\omega_k}\right)^{1/2}$$

decay instability is again realized. For smaller values of p the second condition (3.4) fails. Then Eq. (3.5) is reduced to the form

$$\Omega^2(\Omega - pV\cos\theta - Vp^2/2k) = p^2s^2q'A^2 \tag{3.9}$$

The instability is a maximum on the surface

$$\cos\theta = -\frac{p}{2k_0}, \quad \operatorname{Im}\Omega = \frac{\sqrt{3}}{2}(p^2s^2q'A^2)^{1/3} \tag{3.10}$$

Differentiating (3.9) with respect to $\mathbf{p}$, we find up to small terms

$$\frac{\partial\Omega}{\partial\mathbf{p}} = \frac{1}{3}\mathbf{V} \tag{3.11}$$

We refer to (3.10) as a modified instability. For

$$p/k \sim \left(\frac{s}{V}\frac{q'A^2}{\omega_k}\right)^{1/4}$$

its growth rate is comparable with Lp^2. For smaller values of p/k Eq. (3.9) must be replaced by the expression

$$\Omega^2(\Omega - pV\cos\theta)^2 = k^{-1}Vp^4s^2q'A^2 \tag{3.12}$$

The maximum growth rate

$$\operatorname{Im}\Omega = p\,(Vs^2q'A^2/k^{-1})^{1/4} \tag{3.13}$$

is attained for $\theta = \pi/2$. Differentiating Eq. (3.12) with respect to $\mathbf{p}$, we obtain

$$\partial\Omega/\partial\mathbf{p} = {}^1/_2\mathbf{V} \tag{3.14}$$

3°. $1 \gg q'A^2/\omega_k \gg s/V$. This case differs from the preceding one in that the domain of first-order decay instability is absent, and the modified instability extends to $p \sim k_0$, where it is now required to use Eq. (2.2). The maximum growth rate of the modified instability is

$$\gamma \approx (k_0^2s^2q'A^2)^{1/3} \tag{3.15}$$

We point out that, as implied by Eqs. (3.3), (3.7), and (3.13), for small wave numbers the instability growth rate is proportional to p. This result is directly observable from Eq. (3.5), in which the term $(Lp^2)^2$ can be neglected as $p \to 0$ [20].

Passing to the limit as $p \to 0$ is tantamount to transition to the nonlinear geometric-optical approximation for the HF wave. This approximation was used in [1, 2]. Inclusion of the term $(Lp^2)^2$ is equivalent to the inclusion of Fraunhofer diffraction of HF waves.

As evinced by Eqs. (2.5), (3.8), (3.11), and (3.14), HF wave instabilities due to interaction with LF waves are concomitant in the reference frame of the LF wave and can be excited only for sufficiently long coherent wave packets ($l > V/\operatorname{Im}\Omega$). Mainly an instability with maximum growth rate in the vicinity of $p \sim k_0$ develops here.

For a stationary wave with random phase and coherence length

$$1/p \ll l \ll V/\operatorname{Im}\Omega$$

an instability with $p \sim k_0$ also occurs, but its growth rate is smaller by a factor $l\operatorname{Im}\Omega/V$. This is not true, however, for the "long-wave" instabilities (3.7) and (3.13), whose growth rate is independent of the longitudinal phase structure of the wave, because the geometric-optical approximation is valid for its description (the "Landau damping" of long-wave perturbations by quanta of the primary wave is eliminated by the concomitant character of the instability). The development of long-wave instabilities results in the growth of large-scale inhomogeneities of the medium extending in the direction of the propagation axis; the curvature of the ray paths at these inhomogeneities leads to the formation of caustics and an abrupt increase in the HF amplitude near them. Nonlinear dissipative mechanisms can be activated in this case, and damping of the HF waves can take place in a transparent medium. The further growth of the instability can also lead to the formation of transverse shock waves and the development of two-dimensional "acoustic" [21] turbulence of the medium. For short wave packets with $l \ll V/\mathrm{Im}\,\Omega$ LF instabilities do not develop in general.

We now assess the influence of the intrinsic HF nonlinearity. For a monochromatic wave or a long packet it is required to compare the maximum growth rate of the intrinsic modulation instability with the decay (for $q'A/\omega_k \ll s/V$) or modified-decay (for $q'A^2/\omega_k > s/V$) instability. We find that under the conditions

$$\frac{q'A^2}{\omega_k} \ll \left(\frac{q'}{q}\right)^2 \frac{s}{V} \quad \text{for} \quad q > q' \tag{3.16}$$
$$\frac{q'A^2}{\omega_k} \ll \left(\frac{q'}{q}\right)^{3/2} \frac{s}{V} \quad \text{for} \quad q < q'$$

mainly a LF instability will develop. In the opposite case the influence of the LF waves can be neglected. Note that under conditions (3.16) a modulation instability can develop for sufficiently short packets satisfying the condition

$$\frac{1}{k_0}\left(\frac{\omega_k}{qA^2}\right)^{1/2} < l < \frac{V}{\mathrm{Im}\,\Omega}$$

For a stationary wave with random phase a special analysis is required in order to compare the intrinsic and LF instabilities.

4. Instability of Electromagnetic Waves

We now apply the foregoing results to the instability analysis of electromagnetic waves in a plasma and a nonlinear dielectric.

We represent the electric field of the wave in the form

$$\mathbf{E} = \frac{1}{2}\left(\mathrm{S}e^{-i\omega t + i\mathbf{kr}} + \mathrm{S}^* e^{i\omega t - i\mathbf{kr}}\right)$$

Then the energy of the field has the form

$$\mathrm{E} = \frac{1}{8\pi}\int \frac{1}{\omega_\mathbf{k}} \frac{\partial}{\partial\omega}(\varepsilon\omega^2)\, \mathrm{S_k S_k^*}\, d\mathbf{k}$$
$$\mathrm{S_k} = \frac{1}{(2\pi)^{3/2}}\int \mathrm{S}(r)\, e^{i\mathbf{kr}}\, d\mathbf{r}$$

The energy is expressed in terms of canonical variables according to the relation

$$\mathrm{E} = \int \omega_\mathbf{k} a_\mathbf{k} a_\mathbf{k}^*\, d\mathbf{k}$$

whence

$$\mathbf{a_k} = \frac{1}{\omega_\mathbf{k}}\sqrt{\frac{1}{8\pi}\frac{\partial}{\partial\omega}(\varepsilon\omega^2)}\, \mathrm{S_k}$$

We assume for simplicity that the wave has linear polarization. In this case a_k can be regarded as a scalar, and Eqs. (1.9)-(1.11) are directly applicable, with

$$\Psi = \frac{1}{\omega_k}\sqrt{\frac{1}{8\pi}\frac{\partial}{\partial\omega}(\omega^2\varepsilon)}\, S$$

The quantity qA^2 represents a nonlinear frequency shift in a monochromatic wave:

$$qA^2 = \Delta\omega = \omega\delta n_{nl} n_0^{-1}$$

Here δn_{nl} is the nonlinear correction to the refractive index. Similarly,

$$q'A^2 = \omega\delta n_s n_0^{-1}$$

where δn_s is the striction correction to the refractive index. In a nonlinear dielectric a first-order decay instability represents stimulated Mandel'shtam–Brillouin scattering and has a very low threshold. The parameter s/V is very small for a dielectric ($\sim 10^{-5}$), so that even for the most realistic field amplitudes ($\delta n_s/n_0 \sim 10^{-5}$) the decay instability goes over to a modified instability. The ratio q'/q fluctuates for a dielectric between the limits 0.1 to 10. For $q' \lessdot q$ the criterion (3.16) is also readily satisfied, so that it becomes possible to observe modulation instability in long coherent packets. As for the observation of modulation instability in short pulses, this effect should be expected only in media having a sufficiently strong noninertial nonlinearity.

Ion-acoustic oscillations exist in a plasma without a magnetic field only for $T_e \gg T_i$. For the quantities s/V and $q'A^2$ we have

$$\frac{s}{V} = \left(\frac{m}{M}\right)^{1/2} \frac{\omega_k}{kc} \frac{V_{Te}}{c}, \quad \omega^2 = \omega_p^2 + k^2c^2$$
$$\frac{q'A^2}{\omega_k} = \frac{\omega_p^4}{8\omega_k^4} \frac{c^2}{V_{Te}^2} \frac{E^2}{4\pi m n_0 c^2}$$

The quantity qA^2, which coincides with the maximum growth rate of the intrinsic instability, has been calculated in [20]:

$$qA^2 = \frac{\omega_p^4}{\omega_k^3}\left(\frac{3}{4} - \frac{k^2c^2}{3\omega_p^2 + 4k^2c^2}\right)\frac{E^2}{4\pi m n_0 c^2}$$
$$q / q' \sim V_{Te}^2 / c^2 \ll 1$$

The analysis in that paper is valid in a plasma for field amplitudes satisfying the condition

$$\frac{E^2}{4\pi m n_0 c^2} \ll \frac{\omega^2}{\omega_p^2} \frac{V_{Te}^2}{c^2}$$

as otherwise the oscillational velocity of the electrons is greater than their thermal velocity and it is required to include nonlinear corrections in the ion-acoustic dispersion law.

Consequently, for long coherent wave packets in not too hot a plasma, predominantly decay (or modified) instability develops. Intrinsic instability due to electron nonlinearity can develop for short packets, and for a stationary nonmonochromatic wave any one of the long-wave instabilities can occur.

In an isothermal plasma, $T_e \sim T_i$, sufficiently small-amplitude waves experience only intrinsic instability. However,

$$\frac{E^2}{4\pi m n_0 c^2} \gg \frac{m 8\omega^5 V_{Te}^3}{M \omega_p^4 k c^4} \quad \text{for } q'A^2 / \omega_k \ll \frac{s^2}{V^2}$$

and for low frequencies the LF instability growth rate exceeds the ion-acoustic frequency. For waves of this amplitude instability occurs for any temperature ratio in the plasma.

5. Waves in Weakly Dispersive Media

Considerable importance attaches to the investigation of nonlinear waves in media characterized by weak dispersion, in which

$$\omega_k = sk\,(1 + \lambda k^2) \qquad (\lambda k^2 \ll 1)$$

This dispersion law occurs, for example, for waves on the surface of shallow water or for ion-acoustic waves in a plasma. Media with weak dispersion are described by the hydrodynamical equations with an additional term (see, e.g., [18]):

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \nabla \Phi = 0$$
$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla \Phi)^2}{2} = -\frac{s^2}{\rho_0}\left(\delta\rho + \frac{3}{2} g \frac{\delta\rho^2}{\rho_0} - 2\lambda \Delta \delta\rho\right) \tag{5.1}$$

Here Φ is the hydrodynamic potential, and the variables $\delta\rho$ and Φ are canonically conjugate. Introducing the variables a_k according to Eq. (1.6), we obtain the interaction Hamiltonian in the form

$$H_i = \int [V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a_\mathbf{k} a_{\mathbf{k}_1} a_{\mathbf{k}_2}^* + (\ )^*]\, \delta_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2}\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2 + \frac{1}{3}\int [U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a_\mathbf{k} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + (\ )^*]\, \delta_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2}\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2 \tag{5.2}$$

$$V_{\mathbf{kk}_1\mathbf{k}_2} = U_{\mathbf{kk}_1\mathbf{k}_2} = \frac{s^{1/2}}{16\,(\pi^3\rho_0)^{1/2}}\left\{\frac{(\mathbf{kk}_1)\,k_2^{1/2}}{k^{1/2}k_1^{1/2}} + \frac{(\mathbf{kk}_2)k_1^{1/2}}{k^{1/2}k_2^{1/2}} + \frac{(\mathbf{k}_1\mathbf{k}_2)\,k^{1/2}}{k_1^{1/2}k_2^{1/2}} + 3g\,(kk_1k_2)^{1/2}\right\}$$

and the equations for $a_{\mathbf{k}}$ are

$$\frac{\partial a_{\mathbf{k}}}{\partial t} + i\omega_{\mathbf{k}} a_{\mathbf{k}} = -\,i\int\left\{V_{\mathbf{kk}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} + 2V^*_{\mathbf{kk}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2}^* \delta_{\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2} + U^*_{\mathbf{kk}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2}\right\} d\mathbf{k}_1\, d\mathbf{k}_2 \tag{5.3}$$

Equations (5.1) describe stationary nonlinear waves, which are almost sinusoidal if the nonlinear frequency shift qA^2 is much smaller than the dispersion frequency correction. This condition is expressed by the inequality

$$qA^2/\omega_k \ll \lambda k_0^2$$

which is the condition for applicability of the results of this section. Here k_0 is the stationary wave number.

We consider the instability problem for a stationary wave initially with respect to long-wave perturbations with $p \ll k_0$.

In this case the waves represent a packet narrow in k-space with a spread of the order p. Due to nonlinear interaction, waves with other numbers also occur, so that $a_{\mathbf{k}}$ can be approximately represented in the form

$$a_{\mathbf{k}} = a_{\mathbf{k}}^{(0)} + a_{\mathbf{k}}^- + a_{\mathbf{k}}^+ + b_{\mathbf{k}}^\circ \tag{5.4}$$

Here $a_{\mathbf{k}}^{(0)}$ is the fundamental wave packet with center frequency $\omega(\mathbf{k}_0)$, the $a_{\mathbf{k}}^{\pm}$ are concentrated near $\mathbf{k} = \pm 2\mathbf{k}_0$ and have center frequencies $\pm 2\omega(\mathbf{k}_0)$, and $b_{\mathbf{k}}^0$ represents the "intrinsic" LF field component of $a_{\mathbf{k}}$ concentrated in the interval of wave numbers $\mathbf{k} \sim \mathbf{p}$ and having a frequency $\Delta\omega \sim \omega(\mathbf{k}_0 + \mathbf{p}) - \omega(\mathbf{k}_0)$. On the right-hand sides of the equations for $a_{\mathbf{k}}^{\pm}$ and $b_{\mathbf{k}}^0$ we include only squared terms in $a_{\mathbf{k}}^{(0)}$.

We obtain approximately

$$\begin{aligned}\frac{\partial a_{\mathbf{k}}^+}{\partial t} + i\omega_{\mathbf{k}} a_{\mathbf{k}}^+ &= -\,iV_{2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}\int a_{\mathbf{k}_1}^{(0)} a_{\mathbf{k}_2}^{(0)} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\, d\mathbf{k}_1\, d\mathbf{k}_2 \\ \frac{\partial a_{\mathbf{k}}^-}{\partial t} + i\omega_{\mathbf{k}} a_{\mathbf{k}}^- &= -\,iU^*_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}\int a_{\mathbf{k}_1}^{(0)*} a_{\mathbf{k}_2}^{(0)*} \delta_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2}\, d\mathbf{k}_1\, d\mathbf{k}_2\end{aligned} \tag{5.5}$$

$$\frac{\partial b_{\mathbf{k}}^\circ}{\partial t} + i\omega_{\mathbf{k}} b_{\mathbf{k}}^\circ = -\,2i\int V_{\mathbf{kk}_0\mathbf{k}_0} a_{\mathbf{k}_1}^{(0)} a_{\mathbf{k}_2}^{(0)*} \delta_{\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2}\, d\mathbf{k}_1\, d\mathbf{k}_2 \tag{5.6}$$

Equations (5.5) can be solved explicitly:

$$\begin{aligned}a_{\mathbf{k}}^+ &= -\,\frac{V_{2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}}{\omega(2\mathbf{k}_0) - 2\omega(\mathbf{k}_0)}\int a_{\mathbf{k}_1}^{(0)} a_{\mathbf{k}_2}^{(0)} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\, d\mathbf{k}_1\, d\mathbf{k}_2 \\ a_{\mathbf{k}}^- &= -\,\frac{U^*_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}}{\omega(2\mathbf{k}_0) + 2\omega(\mathbf{k}_0)}\int a_{\mathbf{k}_1}^{(0)*} a_{\mathbf{k}_2}^{(0)*} \delta_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2}\, d\mathbf{k}_1 d\mathbf{k}_2\end{aligned} \tag{5.7}$$

Equation (5.6) coincides with Eq. (1.4) for the external LF component, so that $b_{\mathbf{k}}^\circ$ can be regarded as independent LF degrees of freedom and the results of the preceding part of the article applied to it. In particular, we have

$$f(\mathbf{k}, \mathbf{k}_0) = 2V_{\mathbf{kk}_0\mathbf{k}_0}, \quad \alpha = k_0, \quad \beta = \frac{(3g+1)\,sk_0}{2\rho_0}$$

The quantities $a_{\mathbf{k}}^{\pm}$, upon substitution into Eq. (5.3), yield the intrinsic HF nonlinearity, where

$$q = -\,\frac{|V_{2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}|^2}{\omega(2\mathbf{k}_0) - 2\omega(\mathbf{k}_0)} - \frac{|U_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}|^2}{\omega(2\mathbf{k}_0) + 2\omega(\mathbf{k}_0)} \approx -\,\frac{3\,(g+1)^2}{16\lambda\rho_0}$$

We note that $q'/q \sim \lambda k_0^2 \ll 1$. Inasmuch as $q' \ll q$, stationary self-focusing is determined by the sign of q and occurs in media for which $q < 0$ and, accordingly, $\lambda > 0$. In the case $\lambda > 0$ first-order decay instability is possible, while second-order decay instability is forbidden (for $p \sim k_0$). Conversely, for $\lambda < 0$ first-order decay instability is forbidden, and second-order instability is allowed. We introduce the quantity q_{eff} according to Eq. (3.2). We have

$$q_{\mathrm{eff}} = q\left(1 + \frac{6\lambda k_0^2}{1 - \cos\theta - 3\lambda k_0^2 \cos\theta}\right) \tag{5.8}$$

The quantity q_{eff} is negative for almost all angles except in the narrow cone

$$\theta^2 \sim 6\lambda k_0^2$$

For all angles outside this cone we can use Eq. (3.2); modulation instability is possible here with a growth rate $\sim qA^2$. Near angles $\theta^2 \sim 6\lambda k_0^2$ a stronger instability occurs, representing a modification of first-order decay instability and going over to the latter for $p/k_0 > qA^2/\omega_k \lambda k_0^2$.

The growth rate of this instability attains a maximum for $p \sim k_0$ and is equal to $\gamma \sim (qA^2\lambda k_0^2 s k_0)^{1/2}$.

In the case $\lambda < 0$ we can use Eq. (2.1) for all angles, where the quantity $Lq_{\mathrm{eff}} > 0$ over the entire range of angles, and modulation instability is absent. The latter situation, which is caused by the coincidence of zeros of the functions $L(\theta)$ and $q_{\mathrm{eff}}(\theta)$, is true only up to terms $\lambda k_0^2 (p/k)^{1/3}$.

We also note that

$$Lq_{\mathrm{eff}} = {}^9/_8\,(g+1)^2 s k_0 > 0 \quad \text{for} \quad \theta = 0$$

Consequently, wave instability is absent in weakly dispersive media in the one-dimensional problem, regardless of the sign of λ.

For

$$p/k \sim \left(\frac{qA^2}{\omega_k \lambda k_0^2}\right)^{1/3}$$

a second-order decay instability is found. To calculate its growth rate we perform a canonical transformation to the Hamiltonian (1.3) (see [7]):

$$a_{\mathbf{k}} = c_{\mathbf{k}} - \int \frac{V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} c_{\mathbf{k}_1} c_{\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\, d\mathbf{k}_1 d\mathbf{k}_2}{\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}} + 2\int \frac{V^*_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}} c_{\mathbf{k}_1} c_{\mathbf{k}_2}}{\omega_{k_2} - \omega_k - \omega_{k_1}} \delta_{\mathbf{k}_2-\mathbf{k}-\mathbf{k}_1}\, d\mathbf{k}_1\, d\mathbf{k}_2 - \int \frac{U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} c^*_{\mathbf{k}_1} c^*_{\mathbf{k}_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}} \delta_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2}\, d\mathbf{k}_1\, d\mathbf{k}_2$$

Following this transformation the effective interaction Hamiltonian

$$H_i = \int W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\, c_{\mathbf{k}}^*\, c_{\mathbf{k}_1}^*\, c_{\mathbf{k}_3}\, c_{\mathbf{k}_2}\, \delta_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3 \tag{5.9}$$

is obtained, where

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = -\frac{U_{-(\mathbf{k}_2+\mathbf{k}_3)\mathbf{k}_2\mathbf{k}_3} U^*_{-(\mathbf{k}+\mathbf{k}_1)\,\mathbf{k}\mathbf{k}_1}}{\omega_{\mathbf{k}+\mathbf{k}_3} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3}} - \frac{U^*_{-(\mathbf{k}+\mathbf{k}_1)\,\mathbf{k}\mathbf{k}_2} U_{-(\mathbf{k}_2+\mathbf{k}_3)\,\mathbf{k}_2\mathbf{k}_3}}{\omega_{\mathbf{k}+\mathbf{k}_1} + \omega_{\mathbf{k}} + \omega_{\mathbf{k}_1}} - \frac{V^*_{(\mathbf{k}_2+\mathbf{k}_3)\,\mathbf{k}_2\mathbf{k}_3} V_{\mathbf{k}+\mathbf{k}_1\mathbf{k}\mathbf{k}_1}}{\omega_{\mathbf{k}_2+\mathbf{k}_3} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}} - \frac{V^*_{\mathbf{k}+\mathbf{k}_1\mathbf{k}\mathbf{k}_1} V_{\mathbf{k}_2+\mathbf{k}_3\mathbf{k}_2\mathbf{k}_3}}{\omega_{\mathbf{k}+\mathbf{k}_1} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}_1}} -$$

$$- \frac{2V_{\mathbf{k}\mathbf{k}_2\mathbf{k}-\mathbf{k}_2} V^*_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_3-\mathbf{k}_1}}{\omega_{\mathbf{k}_3-\mathbf{k}_1} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_3}} - \frac{2V_{\mathbf{k}_1\mathbf{k}_3\mathbf{k}_1-\mathbf{k}_3} V^*_{\mathbf{k}_2\mathbf{k}\mathbf{k}_2-\mathbf{k}}}{\omega_{\mathbf{k}_2-\mathbf{k}} + \omega_{\mathbf{k}} - \omega_{\mathbf{k}_2}} - \frac{2V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_1-\mathbf{k}_2} V^*_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_3-\mathbf{k}}}{\omega_{\mathbf{k}_3-\mathbf{k}} + \omega_{\mathbf{k}} - \omega_{\mathbf{k}_3}} - 2\,\frac{V_{\mathbf{k}\mathbf{k}_3\mathbf{k}-\mathbf{k}_3} V^*_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_2-\mathbf{k}_1}}{\omega_{\mathbf{k}_2-\mathbf{k}_1} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}} \tag{5.10}$$

and for $\lambda < 0$ the denominator in (5.10) is nonvanishing.

The dispersion relation can be deduced from Eq. (2.2), in which it is required to put $\Gamma_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = 0$ (see also [7]). We then have

$$\operatorname{Im}\Omega = {}^1/_2 \sqrt{(\omega_{\mathbf{k}_0+\mathbf{p}} + \omega_{\mathbf{k}_0-\mathbf{p}} - 2\omega_{\mathbf{k}_0} + 2A^2 T(\mathbf{p}, \mathbf{k}_0))^2 - 4F^2(\mathbf{p}, \mathbf{k}_0) A^4} \tag{5.11}$$

where

$$F(\mathbf{p}, \mathbf{k}_0) = W_{\mathbf{k}_0+\mathbf{p},\, \mathbf{k}_0-\mathbf{p},\, \mathbf{k}_0\mathbf{k}_0}\; F(0, \mathbf{k}_0) = q$$
$$G(\mathbf{p}, \mathbf{k}_0) = W_{\mathbf{k}_0+\mathbf{p},\, \mathbf{k}_0,\, \mathbf{k}_0+\mathbf{p},\, \mathbf{k}_0},\; T(\mathbf{p}, \mathbf{k}_0) = G(\mathbf{p}, \mathbf{k}_0) + G(-\mathbf{p}, \mathbf{k}_0) + q$$

Indeterminacies of the type

$$\lim_{\varepsilon \to 0} \frac{|V(\varepsilon, \mathbf{k}_0, \mathbf{k}_0)|^2}{\omega(\varepsilon) + \omega(\mathbf{k}_0 + \varepsilon) - \omega(\mathbf{k}_0)}$$

occur in the calculation of q and G, and for their resolution we note that expression (5.10) must go over to (3.2) for $p \ll k_0$, with q_{eff} evaluated according to Eq. (5.8). We infer from a comparison of these expressions that all indeterminate terms are to be set equal to zero.

For not too small values of p/k we can assume on the basis of (5.11) that

$$(\operatorname{Im}\Omega)_{\max} \approx 4\,|F(\mathbf{p}, \mathbf{k}_0)|\,A^2$$

where p is calculated on the surface $2\omega_{k_0} = \omega_{k_0+p} + \omega_{k_0-p}$. For $F(p, k_0)$ we have approximately

$$F(\mathbf{p}, \mathbf{k}_0) = \frac{-2\,|V_{2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}|}{\omega(2\mathbf{k}_0) - 2\omega(\mathbf{k}_0)} - \frac{4\,|V_{\mathbf{k}_0+\mathbf{p},\,\mathbf{k}_0,\,\mathbf{p}} V^{*}_{\mathbf{k}_0,\,\mathbf{k}_0-\mathbf{p},\,\mathbf{p}}|}{\omega(\mathbf{p}) + \omega(\mathbf{k}_0 - \mathbf{p}) - \omega(\mathbf{k}_0)} \tag{5.12}$$

From (5.10) we reject terms whose demoninators do not have an order of smallness equal to λk^2_0.

Adding an orthogonal increment δ to p, we obtain the resonance surface equation

$$\delta^2 = 6p^2(k^2 - p^2)$$

All vectors p lying on the resonance surface are almost parallel to k_0. In (5.2), therefore, we can replace the scalar products by the products of the moduli. We finally have (up to λk^2_0)

$$F(p, k_0) = q\left(1 - \frac{k_0(k_0^2 - p^2)^{1/2}}{k_0^2 + p^2}\right)$$

The maximum growth rate

$$\gamma_{\max} = qA^2 = \Delta\omega_{nl}$$

is attained by the instability at the "upper end" of the resonance surface as $p \to k_0$.

An analogous procedure for the introduction of the intrinsic LF degrees of freedom and resolution of indeterminacies is applicable to other media having a square-law nonlinearity. An exception is the case in which

$$\lim_{\varepsilon \to 0} \frac{|V(\varepsilon, \mathbf{k}_0, \mathbf{k}_0)|^2}{\omega(\varepsilon) + \omega(\mathbf{k}_0 + \varepsilon) - \omega(k_0)} = 0$$

In this case there is no indeterminacy, the influence of the intrinsic LF component can be disregarded, and the interaction of HF waves can be described by means of the effective Hamiltonian (1.3).

In conclusion the authors wish to thank R. Z. Sagdeev for a discussion of the results.

LITERATURE CITED

1. A. A. Vedenov and L. I. Rudakov, "Wave interaction in continuous media," Dokl. Akad. Nauk SSSR, 159, No. 4 (1964).
2. A. A. Vedenov, A. V. Gordeev, and L. I. Rudakov, "Oscillation and instability of a weakly turbulent plasma," Plasma Phys., 9, No. 6 (1967).
3. A. S. Bakai, "Interaction of high-frequency and low-frequency waves in nonlinear dispersive media," Zh. Éksp. Teor. Fiz., 55, No. 1 (1968).
4. V. E. Zakharov, "Hamiltonian formalism for hydrodynamic models of a plasma," Zh. Éksp. Teor. Fiz., 60, No. 5 (1971).
5. V. E. Zakharov and E. A. Kuznetsov, "The variational principle and canonical variables in magnetohydrodynamics," Dokl. Akad. Nauk SSSR, 194, No. 6 (1970).
6. V. E. Zakharov, "Stability of finite-amplitude periodic waves on the surface of a deep liquid," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 2 (1968).
7. V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, "Instability of monochromatic spin waves," Fiz. Tverd. Tela, 11, No. 10 (1969).
8. É. Shleman, "Growth of spin waves in parametric excitation in the transition regime," Izv. Akad. Nauk SSSR, Ser. Fiz., 28, No. 3 (1964).
9. V. N. Oraevskii and R. Z. Sagdeev, "Stability of steady-state longitudinal plasma oscillations," Zh. Tekh. Fiz., 32, No. 11 (1962).
10. N. Bloembergen, Nonlinear Optics, W. A. Benjamin, Menlo Park, Calif. (1965).
11. I. L. Fabelinskii, Molecular Scattering of Light [in Russian], Nauka, Moscow (1965).
12. T. F. Volkov, "Influence of a high-frequency electromagnetic field on plasma oscillations," in: Plasma Physics and the Problem of Controlled Thermonuclear Reactions [in Russian], Vol. 4, Izd. Akad. Nauk SSSR, Moscow (1958), p. 98.
13. V. Ts. Gurovich, V. I. Karpman, and R. N. Kaufman, "Stability of solitary electroacoustic waves," Zh. Éksp. Teor. Fiz., 56, No. 6 (1969).
14. K. N. Nishikawa, "Parametric excitation of coupled waves," J. Phys. Soc. Japan, 24, No. 4 (1968).
15. L. M. Gorbunov, "Perturbation of a medium by the field of a strong electromagnetic wave," Zh. Éksp. Teor. Fiz., 55, No. 6 (1968).
16. V. I. Talanov, "Self-focusing of electromagnetic waves in nonlinear media," Izv. VUZ. Radiofizika, 7, No. 3 (1964).

17. Y. R. Shen, "Electrostriction, optical Kerr effect, and self-focusing of laser beam," Phys. Lett., 20, No. 4 (1966).

18. V. E. Zakharov, "Wave stability in nonlinear dispersive media," Zh. Éksp. Teor. Fiz., 51, No. 4 (1966).

19. V. E. Zakharov, V. V. Sobolev, and V. S. Synakh, "Investigation of the behavior of light beams in nonlinear media," Zh. Éksp. Teor. Fiz., 60, No. 1 (1971).

20. A. L. Berkhoer and V. E. Zakharov, "Mutual interaction of waves with different polarizations in nonlinear media," Zh. Éksp. Teor. Fiz., 58, No. 3 (1970).

21. V. E. Zakharov and R. Z. Sagdeev, "Spectrum of acoustic turbulence," Dokl. Akad. Nauk SSSR, 192, No. 2 (1970).